# Recitation 13

## Review

**Inner product:** inner product walks like the dot product, talks like the dot product, and behaves like the dot product. The only difference is, it's defined in a vector space which does not look like $\mathbb{R}^n$.

So **to do things with inner product** you **use the same methods as for dot product**, but replacing dot product by this inner product. For example, here is how the projection formula looks like for a projection of $y \in V$ in some inner product space $V$ onto some subspace $W \subset V$ with fixed **orthogonal** basis $\{u_1, \ldots, u_m\}$:

$$proj_W(y) = \frac{\langle y, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 + \cdots + \frac{\langle y, u_m \rangle}{\langle u_m, u_m \rangle} u_m$$

**Least-squares:** If a system $Ax = b$ is inconsistent, we can find a **least-squares** solution, which is a solution to the equation $A^T A \widehat{x} = A^T b$ (which is called the **normal equation**).

The **idea** is that $\widehat{x}$ is a vector in the domain of $A$ such that $A\widehat{x}$ is as close to $b$ as possible. And so $\widehat{x}$ is "the best" substitute for a solution of $Ax = b$. This $A\widehat{x}$ is just the projection of $b$ onto $Col(A)$.

**Weighted least-squares** solution for $Ax = y$ with weights $W$ (viewed as a diagonal matrix with weights on the diagonal) is **the same** as least square solution for the system $WAx = Wy$, i.e. you really need to solve the system

$$(WA)^T WA\widehat{x} = (WA)^T Wy$$

**Trend analysis and Fourier approximation:** In both cases, you have some set of orthogonal functions, and you are projecting another function to the span of the orthogonal ones. (Be careful with what inner product you are using).

**Orthogonal diagonalization** It is just like the usual diagonalization, but orthogonal. It can only be done to a **symmetric matrix** $A$. Then you do

- Find eigenvalues of $A$.

- Find eigenvectors.

- For each eigenvalue, if you have several lin. independent eigenvectors, orthogonalize them.

- Normalize all the eigenvectors you have. They would be columns of the matrix called $P$.

- **Note:** the order of eigenvectors in $P$ should correspond to the order of eigenvalues, as before.

- Then $D = P^T AP$ is the orthogonal diagonalization of $A$. **Note:** matrix $D$ is the same diagonalization we did a million times before. The main difference is that $P$ is now an orthogonal matrix.

## Problems

**Problem 1.** For the space $C[0, 1]$ of continuous functions on the interval $[0, 1]$ there is an inner product defined by

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt$$

Let $f(t) = t - 2$ and $g(t) = e^t$. Compute $\langle f, g \rangle$.

**Problem 2.** Suppose you and two friends of yours, call them A and B, are making a team for a class project. The project involves measurements of something, and then predicting how this something works. If you want, you can assume that you are trying to model how a magic box showing you random numbers works. You have agreed to do the modeling, and your friends are going to do the measurements.

Unfortunately, professor assigned another person to be on your team, namely, that weird guy named Q whom you know to be a bad guy. Maybe he punched a cat once, or maybe even something worse. But suppose you feel bad to just through away Q's measurements. So what you decide to do is, you are going to weight them half as much as the measurements that your friends did.

You are trying to do the approximation by a line $y = \beta_0 + \beta_1 x$. You friend A tells you that his data is $(x_1, y_1) = (1, 0.5)$. Your friend B gives you his data: it is $(x_2, y_2) = (1.5, 1.5)$. Now Q gives you his measurements. They are $(x_3, y_3) = (2, 3)$ and $(x_4, y_4) = (3, 5)$. You are suspicious: why did Q do two measurements instead of just one? What is he trying to prove? Weird...

Find the weighted least-squares line $y = \beta_0 + \beta_1 x$ approximating the data.

**Problem 3.** Now we are going to do some of that trend analysis that they have now. You've seen it. Suppose we are doing measurements at the points $t = -5, -3, -1, 1, 3, 5$.
Show that the first three orthogonal polynomials are $p_0(t) = 1, p_1(t) = t, p_2(t) = \frac{3}{8}t^2 - \frac{35}{8}$.
Fit a quadratic trend function to the data

$$(-5, 1), \ (-3, 1), \ (-1, 4), \ (1, 4), \ (3, 6), \ (5, 8)$$

**Problem 4.** Find the second order Fourier approximation of the function $f(t) = t - 1$.

**Problem 5.** Orthogonally diagonalize the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

**Problem 6.** Explain why an equation $Ax = b$ has a solution if and only if $b$ is orthogonal to all solutions of the equation $A^T x = 0$.

**Problem 7.** For a quadratic form $Q = 2x_1^2 - 4x_1 x_2 - x_2^2$ on $\mathbb{R}^2$ make a change of variable $x = Py$ that transforms the form into one without cross-product terms.

**Problem 8.** Same question for the quadratic form $Q = x_1^2 - 12x_1 x_2 + 8x_1 x_3 + 2x_2^2 - 4x_2 x_3 - 3x_3^2$. Help: the eigenvalues of the corresponding matrix are $-3, -6, 9$.